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# On a Global Existence Theorem for the Enskog Equation

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In this note we prove global existence of mild solutions of the Enskog equation with small initial data.

**KEY WORDS:** Boltzmann–Enskog equation; hard spheres; mild solutions.

We consider the initial value problem for the Enskog equation for the distribution function of a gas of hard spheres with a diameter $\sigma$:

$$(\partial_t + \xi \cdot \nabla_x) f = Q(f, f) - fRf, \qquad f(0, \cdot, \cdot) = f_0 \tag{1}$$

where

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2_+} \chi \left[ \int_{\mathbb{R}^3} f\left(t, x + \frac{1}{2}\sigma n, v\right) dv \right] \times f(t, x, \xi') f(t, x + \sigma n, \eta') \, d\mu$$

$$Rf = \int_{\mathbb{R}^3} \int_{S^2_+} \chi \int_{\mathbb{R}^3} f\left(t, x - \frac{1}{2}\sigma n, v\right) dv \Bigg] \times f(x, x - \sigma n, \eta) \, d\mu$$

$n$ is the unit vector joining the centers of the colliding particles, $\xi$, $\eta$, $\xi'$, $\eta'$, $v$, $x \in \mathbb{R}^3$, $g := \xi - \eta$, $\xi' := \xi + n(gn)$, $\eta' := \eta - n(gn)$, $S^2_+ := \{n \in \mathbb{R}^3 \colon |n| = 1, n(\xi - \eta) \geqslant 0\}$, $d\mu := ng\,dn\,d\eta$, and $\chi$ is the Enskog correlation function.[1] Existence of local nonnegative solutions of (1) for nonnegative initial data

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was established in Ref. 2. Our objective here is to prove global existence for (1). Since solutions of (1) are in their intervals of existence, bounded by the solutions of the following truncated initial value problem:

$$(\partial_t + \xi \cdot \nabla_x) f = Q(f, f), \qquad f(0, \cdot, \cdot) = f_0 \tag{2}$$

it is enough to prove global existence for (2), which in mild form reads

$$f^{\#}(t, x, \xi) = f_0(x, \xi) + \int_0^t \int_{\mathbb{R}^3} \int_{S_+^2} f^{\#}(\tau, z, \xi') f^{\#}(\tau, y, \eta') \times \chi \left[ \int f^{\#}\left(\tau, x + \frac{1}{2}\sigma n, v\right) dv \right] d\mu \tag{3}$$

where

$$f^{\#}(t, x, \xi) := f(t, x + \xi t, \xi)$$
$$z := x + \tau(\xi - \xi')$$
$$y := x + \sigma n + \tau(\xi - \eta')$$

We define, for $\beta > 0$, the following functional spaces: $S_\beta$ is the space of completions of the $C_0$ functions with respect to the norm:

$$\|f_0\|_\beta = \int_{\mathbb{R}^3} \sup_{x \in \mathbb{R}^3, n \in S^3} \exp(\beta |x - n\sigma|^2) f_0(x, \xi)| \, d\xi \tag{4}$$

and

$$C_\beta = \{f^{\#} \in C([0, \infty); S_\beta) : f^{\#}(0, \cdot, \cdot) = f_0\}$$

with the norm

$$|||f|||_\beta = \int_{\mathbb{R}^3} \sup_{t \in [0,T), x \in \mathbb{R}^3, n \in S^2} \exp(\beta |x - n\sigma|^2) |f^{\#}(t, x, \xi)| \, d\xi \tag{5}$$

In order to prove global existence for (3), we apply a contraction mapping argument with the operator $U: C_\beta^+ \to C_\beta^+$ defined by

$$Uf^{\#}(t, x, \xi) := f_0(x, \xi) + \int_0^t Q^{\#}(f, f)(\tau, x, \xi) \, d\tau \tag{6}$$

where $Q^{\#}$ is given by the corresponding expression on the right-hand side of (3), and $C_\beta^+$ denotes the set of the nonnegative functions in $C_\beta$.

We prove the following:

**Theorem.** Let the following conditions be satisfied:

$$C_1 := \sup \left\{ \chi \left[ \int f^{\#}\left(t, x + \frac{1}{2}\sigma n, v\right) dv \right] \exp(-2\beta\sigma\bar{n}x) \right.$$
$$\left. x \in \mathbb{R}^3; n, \bar{n} \in S^2; t \in [0, T); f^{\#} \in C_\beta^+ \right\} < \infty \tag{7}$$

There exists a function $X_L: \mathbb{R}^3 \to \mathbb{R}_+$ such that for $f^{\#}, g^{\#} \in C_\beta^+$

$$|f^{\#}(Z)\,\chi[f^{\#}(X)] - g^{\#}(Z)\,\chi[g^{\#}(X)]| \leqslant X_L(x)\,|f^{\#}(Z) - g^{\#}(Z)| \tag{8}$$
$$x, z, \xi \in \mathbb{R}^3; \qquad \tau \in [0, T)$$

where we denoted $f^{\#}(Z) := f^{\#}(\tau, z, \xi')$ and

$$\chi[f^{\#}(X)] := \chi\left[\int f^{\#}\left(\tau, x + \frac{1}{2}\sigma n, v\right) dv\right]$$

The function $X_L$ is assumed to satisfy the decay condition, similar to (7),

$$C_2 := \sup\{X_L(x)\exp(-2\beta\sigma n x), x \in \mathbb{R}^3, n \in S^2\} < \infty \tag{7'}$$

Then $U$ is a contraction on

$$B_C := \{f^{\#} \in C_\beta^+ : |||f|||_\beta \leqslant 1/C\}$$

for $\|f_0\|_\beta \leqslant 1/2C$, where $C > 0$ is a constant.

*Remark.* The assumptions (7) for the Enskog correlation function can be interpreted as decay conditions outside arbitrary large bounded regions in physical space. Recently the intial value problem for (1) was treated in Ref. 4 in a different Banach space setting.

*Proof.* We note the identity

$$|z|^2 + |y - n\sigma|^2 = |x|^2 + |x + \tau(\xi - \eta)|^2 \tag{9}$$

Such an identity was used, for $\sigma = 0$, by Illner and Shinbrot[3] for the Boltzmann equation.

Multiplying (6) by $\exp(\beta|x - n\sigma|^2)$ and using (7) and (9), we obtain (where $C$ denotes different constants)

$$Uf^{\#}(t, x, \xi)\exp(\beta|x - \sigma n|^2)$$
$$\leqslant f_0(x, \xi)\exp[\beta|x - \sigma n|^2] + C\iiint f^{\#}(\tau, z, \xi')[\exp(\beta|z|^2)]\, f^{\#}(\tau, y, \eta')$$
$$\times [\exp(\beta|y - \sigma n|^2)\exp(-\beta|x + \tau(\xi - \eta)|^2)]\; |\xi - \eta|\; dn\, d\eta\, d\tau \tag{10}$$

Now we estimate the functions of $y$ by the corresponding suprema, take the suprema over $x$, $n$, and $t \in [0, T)$, and integrate finally over $d\xi$, obtaining

$$|||Uf^{\#}|||_{\beta} \leqslant |||f_0|||_{\beta} + C|||f|||_{\beta}^2 \tag{11}$$

i.e., $U$ maps a ball in $C^+$ into itself for $\|f_0\|_{\beta}$ small enough. Similarly,

$$|||Uf^{\#} - Ug^{\#}|||_{\beta} \leqslant C(|||f^{\#}|||_{\beta} + |||g^{\#}|||_{\beta})\, |||f^{\#} - g^{\#}|||_{\beta} \tag{12}$$

(11) and (12) imply the statement of the theorem. Global existence for (1) follows from the Banach fixed point theorem.

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